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Schur Function Expansions and the Rational Shuffle Conjecture

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Abstract. Gorsky and Negut introduced operators $Q_{m,n}$ on symmetric functions and conjectured that, in the case where m and n are relatively prime, the expansion of $Q_{m,n}(-1)^n$ in terms of the fundamental quasi-symmetric functions are given by polynomials introduced by Hikita. Later Bergeron, Garsia, Leven, and Xin extended and refined the conjectures of Gorsky and Negut to give a combinatorial interpretation of the coefficients that arise in expansion of $Q_{m,n}(-1)^n$ in terms of the fundamental quasi-symmetric functions for arbitrary m and n which we will call the rational shuffle conjecture. The rational shuffle conjecture was later proved by Mellit in 2016. The main goal of this paper is to study the combinatorics of the coefficients that arise in the Schur function expansion of $Q_{m,n}(-1)^n$ in the case where m or n equals 3.

Résumé. Gorsky et Negut opérateurs introduit $Q_{m,n}$ sur les fonctions symétriques et l'hypothèse que, dans le cas où m et n sont relativement premier, l'expansion de $Q_{m,n}(-1)^n$ en termes de la quasi-fondamental fonctions symétriques sont données par des polynômes introduites par Hikita. Plus tard, Bergeron, Garsia, Leven, et Xin étendu et affiné les conjectures de Gorsky et Negut pour donner une interprétation combinatoire des coefficients qui surgissent dans l'expansion de $Q_{m,n}(-1)^n$ en termes des fonctions symétriques quasi arbitraire pour m et n que nous appellerons la conjecture rationnelle shuffle. Le rationnel shuffle conjecture a été prouvée par la suite de Mellit en 2016. L'objectif principal de cet article est d'étudier la combinatoire des coefficients qui surgissent dans l'expansion de la fonction Schur $Q_{m,n}(-1)^n$ dans le cas où m ou n est égale à 3.

Keywords: Macdonald polynomials, parking functions, Dyck paths, Shuffle Conjecture

1 Introduction

The rational shuffle conjecture as formulated by Gorsky and Negut [10] and Bergeron, Garsia, Leven, and Xin [3] gives a combinatorial interpretation of the coefficients that arise in the fundamental quasi-symmetric function expansion of certain operators on

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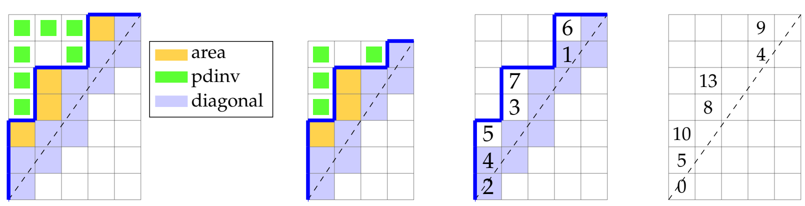
symmetric functions $Q_{m,n}$ applied to $(-1)^n$. This conjecture was proved by Mellit [19]. Leven gave a combinatorial proof of the Schur function expansion of $Q_{2,2n+1}(-1)$ and $Q_{2n+1,2}1$ in [17], and the coefficient at s_{1^n} in $Q_{m,n}(-1)^n$ is known as the rational q,t-Catalan number, computed by Gorsky and Mazin [9] for the case $n = 3$ and studied by Lee, Li and Loehr [16] for the case $n = 4$. The coefficients at the hook-shaped Schur functions were discussed by Armstrong, Loehr and Warrington [1].

In this paper, we explore the combinatorics of the Schur function expansion of $Q_{m,n}(-1)^n$ in the special case where either m or n is less than or equal to 3. In particular, we study the Schur function expansions of $Q_{2,2n}1$, $Q_{2n,2}1$, $Q_{3,n}(-1)^n$, and $Q_{m,3}(-1)$.

To state our results, we must first recall the rational shuffle conjecture. This will require a series of definitions.

Let m and n be positive integers. An (m,n)-Dyck path is a lattice path from $(0,0)$ to (m,n) which always remains weakly above the main diagonal $y = \frac{n}{m}x$. The cells that are cut through by the main diagonal will be called *diagonal* cells. Here Figure 1(a) gives an example of a $(5,7)$-Dyck path, and Figure 1(b) gives an example of a $(4,6)$-Dyck path, where the diagonal cells are the light blue cells.



(a) (b) (c) (d)

Figure 1: A $(5,7)$-Dyck path, a $(4,6)$-Dyck path, a $(5,7)$-parking function and its car ranks

The number of full cells between an (m,n)-Dyck path Π and the main diagonal is denoted $\mathrm{area}(\Pi)$. The collection of cells above a Dyck path Π forms the Ferrers diagram (in English notation) of a partition $\lambda(\Pi)$. In the example of the rational Dyck path pictured in Figure 1(a), $\lambda(\Pi) = (3,3,1,1) =$. Let $\chi(-)$ denotes the function that takes value 1 if its argument is true, and 0 otherwise. The path dinv of an (m,n)-Dyck path Π is given by

$$\mathrm{pdinv}(\Pi) = \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)}{leg(c)+1} \leq \frac{m}{n} < \frac{arm(c)+1}{leg(c)}\right).$$

An (m,n)-*parking function* PF is obtained by labeling the cells east of and adjacent to a north step of an (m,n)-Dyck path with the integers $1,\ldots,n$ in such a way that the numbers increase in each column as we read from bottom to top. We will refer to these labels as *cars*. The underlying Dyck path is denoted as $\Pi(\mathrm{PF})$. Figure 1(c) pictures a $(5,7)$-parking function based on the $(5,7)$-Dyck path pictured in Figure 1(a).

Next we define $\mathrm{dinv}(\mathrm{PF})$ and $\mathrm{ides}(\mathrm{PF})$ for any parking function. We define the rank of a cell (x, y) in the (m, n)-grid to be $\mathrm{rank}(x, y) = my - nx$, Figure 1(d) shows the rank of the cars in Figure 1(c). $\sigma(\mathrm{PF})$, the word of PF, is obtained by reading cars from highest to lowest ranks. In our example, $\sigma(\mathrm{PF}) = 7563412$. We define $\mathrm{ides}(\mathrm{PF})$ to be the descent set of $\sigma(\mathrm{PF})^{-1}$, and we let $\mathrm{tdinv}(\mathrm{PF}) = \sum_{\text{cars } i<j} \chi(\mathrm{rank}(i) < \mathrm{rank}(j) < \mathrm{rank}(i) + m)$. In Figure 1(c), $\mathrm{tdinv}(\mathrm{PF}) = 7$ since the pairs of cars contributing to tdinv are $(1, 3)$, $(1, 4)$, $(3, 5)$, $(3, 6)$, $(4, 6)$, $(5, 7)$ and $(6, 7)$.

Our definition of $\mathrm{dinv}(\mathrm{PF})$ will follow the formulation by Leven and Hicks [14]. Set $\frac{0}{0} = 0$ and $\frac{x}{0} = \infty$ for all $x \neq 0$, then

$$\mathrm{dinvcorr}(\Pi) = \sum_{c\in\lambda(\Pi)} \chi\left(\frac{arm(c)+1}{leg(c)+1} \leq \frac{m}{n} < \frac{arm(c)}{leg(c)}\right) - \sum_{c\in\lambda(\Pi)} \chi\left(\frac{arm(c)}{leg(c)} \leq \frac{m}{n} < \frac{arm(c)+1}{leg(c)+1}\right).$$

Then, for any (m, n)-parking function PF with underlying Dyck path Π, we define

$$\mathrm{dinv}(\mathrm{PF}) = \mathrm{tdinv}(\mathrm{PF}) + \mathrm{dinvcorr}(\Pi).$$

There are alternative definitions of the statistic dinv, see [1, 11].

Given $S \subseteq \{1, 2, \ldots, n-1\}$, let $F_S[X]$ denote the fundamental quasi-symmetric functions of Gessel [8] associated to S, where $X = x_1 + x_2 + \cdots + x_n$. We define the *Hikita polynomial* [15] $\mathrm{H}_{m,n}[X; q, t]$ where m and n are coprime by

$$\mathrm{H}_{m,n}[X; q, t] = \sum_{\mathrm{PF}\in\mathcal{PF}_{m,n}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}[X].$$

For any partition μ of n, let $\widetilde{H}_\mu$ be the modified Macdonald symmetric function [18] associated to μ, and let ∇ be the linear operator defined in terms of the modified Macdonald symmetric functions $\widetilde{H}_\mu(X; q, t)$ by $\nabla \widetilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \widetilde{H}_\mu$, where μ' is its conjugate and $n(\mu) = \sum_i (i-1)\mu_i$. The classical shuffle conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov [12] is that $\nabla e_n = \mathrm{H}_{n+1,n}[X; q, t]$ which was proved by Carlson and Mellit [4].

Gorsky and Negut [10] introduced operators $\mathrm{Q}_{m,n}$ on symmetric functions and conjectured that the $\mathrm{Q}_{m,n}(-1)^n = \mathrm{H}_{m,n}[X; q, t]$ in the case where m and n are coprime. As shown in [3], the $\mathrm{Q}_{m,n}$ operators of the Gorsky-Negut conjecture can be defined in terms of the operators D_k which were introduced in Bergeron and Garsia[2]. If $F[X]$ is a symmetric function and $M = (1-t)(1-q)$, then in plethystic notation,

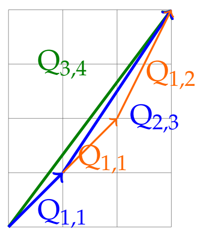
$$D_k\, F[X] = F\left[X + \frac{M}{z}\right] \sum_{i\geq 0} (-z)^i e_i[X]\bigg|_{z^k}.$$

Then one can construct a family of symmetric function operators $\mathrm{Q}_{a,b}$ for any pair of positive integers (a, b) as follows. First for any $n \geq 0$, set $\mathrm{Q}_{1,n} = D_n$. Next, one can recursively define $\mathrm{Q}_{m,n}$ for $m > 1$ as follows. Consider the $m \times n$ lattice with diagonal

$y = \frac{n}{m}x$. Let (a,b) be the lattice point which is closest to and below the diagonal. Set $(c,d) = (m-a, n-b)$. In such a case, we will write $\mathrm{Split}(m,n) = (a,b) + (c,d)$. Then we have the following recursive definition of the $\mathrm{Q}_{m,n}$ operators:

$$\mathrm{Q}_{m,n} = \frac{1}{M}[\mathrm{Q}_{c,d}, \mathrm{Q}_{a,b}] = \frac{1}{M}(\mathrm{Q}_{c,d}\mathrm{Q}_{a,b} - \mathrm{Q}_{a,b}\mathrm{Q}_{c,d}).$$

Figure 2 gives an example of $\mathrm{Split}(3,4)$.



$$\mathrm{Split}(3,4) = (1,1) + (2,3),$$

$$\begin{aligned}\mathrm{Q}_{3,4} &= \tfrac{1}{M}[\mathrm{Q}_{2,3}, \mathrm{Q}_{1,1}] = \tfrac{1}{M}[\mathrm{Q}_{2,3}, D_1] \\ &= \tfrac{1}{M}[\mathrm{Q}_{2,3}, \mathrm{Q}_{1,1}] = \tfrac{1}{M}[\tfrac{1}{M}[D_2, D_1], D_1] \\ &= \tfrac{1}{M^2}(D_2D_1D_1 - 2D_1D_2D_1 + D_1D_1D_2)\end{aligned}$$

Figure 2: The geometry of $\mathrm{Split}(3,4)$

The rational shuffle conjecture of Gorsky and Negut in the case where m and n are relatively prime which was proved by Mellit [19] is the following.

Theorem 1 (Mellit)**.** *If m and n are coprime positive integers, then* $\mathrm{Q}_{m,n}(-1)^n = \mathrm{H}_{m,n}[X;q,t]$.

When m and n are coprime and $k \geq 1$, we defined the return of a (km,kn)-parking function PF, $ret(\mathrm{PF})$ to be the *smallest* positive integer i such that the supporting path of PF goes through the point (im, in). Then Garsia, Leven, Wallach, and Xin [7] conjectured the following theorem which was also proved by Mellit [19].

Theorem 2 (Mellit)**.** *For all pairs of coprime positive integers (m,n) and any $k \in \mathbb{Z}^+$, we have*

$$\mathrm{Q}_{km,kn}(-1)^{kn} = \sum_{\mathrm{PF} \in \mathcal{PF}_{km,kn}} [ret(\mathrm{PF})]_{\frac{1}{t}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}[X].$$

A more important goal is to find the Schur function expansion of ∇e_n since that would allows us to find the character generating function of the ring of diagonal invariants, see [13]. More generally, we would like to find a combinatorial interpretation of of the coefficients that arise in the Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$. The main goal of this paper is to find such Schur function expansions in the case where m or n equals 3. The Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$ in the case where m and n are coprime and either m or n equals 2 was given by Leven [17]. That is, let $[n]_{q,t} = \frac{q^n - t^n}{q-t} = q^{n-1} + q^{n-2}t + \cdots + t^{n-1}$ be the q,t-analogue of n, then Leven proved that for any $k \geq 0$,

$$\mathrm{Q}_{2k+1,2}1 = \mathrm{H}_{2k+1,2}[X;q,t] = [k]_{q,t}s_2 + [k+1]_{q,t}s_{1,1}, \text{ and}$$
$$\mathrm{Q}_{2,2k+1}(-1) = \mathrm{H}_{2,2k+1}[X;q,t] = \textstyle\sum_{r=0}^{k}[k+1-r]_{q,t}s_{2^r1^{2k+1-2r}}.$$

Using the results of Theorem 2, we can prove the following.

Theorem 3. $Q_{2k,2}1 = H_{2k,2}[X;q,t] = ([k]_{q,t} + [k-1]_{q,t})s_2 + ([k+1]_{q,t} + [k]_{q,t})s_{1,1}$ *and*
$Q_{2,2k}1 = H_{2,2k}[X;q,t] = \sum_{r=0}^{k}([k+1-r]_{q,t} + [k-r]_{q,t})s_{2^r1^{2k+1-2r}}$.

The main goal of this paper is to study the combinatorics of the Schur function expansion of $Q_{m,n}(-1)^n$ in the case where either m or $n = 3$. Given that Mellit has proved the rational shuffle conjecture, we can find the Schur function expansion in one of two ways. That is, we can use the properties of the $Q_{m,n}$ to find the Schur function expansion of $Q_{m,n}(-1)^n$ which we will refer to as working on the *symmetric function side* of the rational shuffle conjecture. Second, one could start with the Hikita polynomial $H_{m,n}[X;q,t]$ and expand that polynomial into Schur functions which we will call working on the *combinatorial side* of the rational shuffle conjecture. In the case where $n = 3$, we can prove the following theorem by either working on the symmetric function side or the combinatorial side of the rational shuffle conjecture.

Theorem 4. *For any $k \geq 0$,*

$$\begin{aligned}
Q_{3k+1,3}(-1) = H_{3k+1,3}[X;q,t] &= \left(\textstyle\sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t}\right)s_3\\
&+\left(\textstyle\sum_{i=0}^{k-1}(qt)^{k-1-i}([3i+2]_{q,t}+[3i+3]_{q,t})\right)s_{2,1}+\left(\textstyle\sum_{i=0}^{k}(qt)^{k-i}[3i+1]_{q,t}\right)s_{1^3},\\
Q_{3k+2,3}(-1) = H_{3k+2,3}[X;q,t] &= \left(\textstyle\sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+2]_{q,t}\right)s_3\\
&+\left(\textstyle\sum_{i=0}^{k}(qt)^{k-1-i}([3i]_{q,t}+[3i+1]_{q,t})\right)s_{2,1}+\left(\textstyle\sum_{i=0}^{k}(qt)^{k-i}[3i+2]_{q,t}\right)s_{1^3},\\
Q_{3k,3}(-1) = H_{3k,3}[X;q,t] &= \left(\textstyle\sum_{i=0}^{k-1}(qt)^{k-1-i}\left([3i-1]_{q,t}+[3i]_{q,t}+[3i+1]_{q,t}\right)\right)s_3\\
&+\Big((qt)^{k+1}\left([3]_{q,t}+2[2]_{q,t}+[1]_{q,t}\right)\\
&\quad+\textstyle\sum_{i=1}^{k-1}(qt)^{k-1-i}\left([3i]_{q,t}+2[3i+1]_{q,t}+2[3i+2]_{q,t}+[3i+3]_{q,t}\right)\Big)s_{2,1}\\
&\quad+\left(\textstyle\sum_{i=0}^{k}(qt)^{k-i}([3i-1]_{q,t}+[3i]_{q,t}+[3i+1]_{q,t})\right)s_{1^3}.
\end{aligned}$$

In the case where $m = 3$, we have a conjectured formula but we have not been able to prove it. Nevertheless, by working on the combinatorial side of the rational shuffle conjecture we can prove many remarkable facts about the Schur function expansion of $Q_{3,n}(-1)^n$. Let $[s_\lambda]_{m,n}$ be the coefficient of s_λ in the polynomial $Q_{m,n}(-1)$ and $H_{m,n}[X;q,t]$, then we can combinatorially prove the following facts about $[s_\lambda]_{m,3}$, $[s_\lambda]_{3,n}$ and $[s_\lambda]_{m,n}$.

Theorem 5. **(a)** $[s_{1^3}]_{m-3,3} = [s_3]_{m,3}$, **(b)** $[s_{3^{a+1}2^b1^c}]_{3,n} = [s_{3^a2^b1^c}]_{3,n-3}$, **(c)** $[s_{1^3}]_{n,3} = [s_{1^n}]_{3,n}$, **(d)** $[s_{21}]_{n,3} = [s_{21^{n-2}}]_{3,n}$, *and* **(e)** $[s_{2^a1^b}]_{3,n} = [s_{2^b1^a}]_{3,3(a+b)-n}$.

Next we state a general theorem and two conjectures.

Theorem 6. *For all $m,n > 0$ and $\lambda' \vdash (n-am)$,* **(a)** $[s_{1^n}]_{m-n,n} = [s_n]_{m,n}$, **(b)** $[s_{m^a\lambda'}]_{m,n} = [s_{\lambda'}]_{m,n-am}$, **(c)** $[s_{1^n}]_{m,n} = [s_{1^m}]_{n,m}$, *and* **(d)** $[s_{k1^{n-k}}]_{m,n} = [s_{k1^{m-k}}]_{n,m}$.

Note that Theorem 6(d) is a result about the hook-shaped Schur functions.

Conjecture 1. *Let $a < b$, then* $[s_{2^a1^b}]_{3,n} = \sum_{i=0}^{a}[b+i]_{q,t} + (qt)[s_{2^a1^{b-3}}]_{3,n-3}$.

Conjecture 2. $[s_{(m-1)^{\alpha_{m-1}}(m-2)^{\alpha_{m-2}}\ldots 1^{\alpha_1}}]_{m,n} = [s_{(m-1)^{\alpha_1}(m-2)^{\alpha_2}\ldots 1^{\alpha_{m-1}}}]_{m,(m\sum_{i=1}^{m-1}\alpha_i - n)}$.

The outline of this paper is as follows. First, in Section 2, we shall outline the proof of Theorem 4 from both the symmetric function side and the combinatorial side. The combinatorics involved in proofs from the combinatorial side leads to very interesting combinatorial proofs, but due to lack of space, we can only briefly describe some of the ideas involved. Nevertheless, our combinatorial proofs allow us to prove many general facts of the Schur function expansion of $Q_{m,n}$ in general which are not at all obvious from the symmetric function side. We will exhibit such ideas in Sections 2 to 4.

In Section 3, we shall briefly discuss the proof of Theorem 5. We shall see that the results of Theorem 5 show that the problem of computing the Schur function expansion of $Q_{3,n}(-1)^n$ can be reduced to the problem finding the coefficients of Schur functions of the form $s_{2^a,1^b}$ in $Q_{3,n}(-1)^n$. We will state a conjecture for such coefficients at the end of Section 3. In Section 4, we extend some combinatorial results to $Q_{m,n}(-1)^n$ case.

2 Schur Basis Expansion of $(m, 3)$ Case

2.1 Symmetric Function Side — $Q_{m,3}$(-1)

We need the following lemma from [3] to prove the symmetric function side of the theorem.

Lemma 1. *For any positive m, n, $\nabla Q_{m,n} \nabla^{-1} = Q_{m+n,n}$.*

This allows us to prove the formula by induction:

$$\begin{aligned} Q_{m+3,3}(-1) = \nabla Q_{m,3}\nabla^{-1}(-1) = \nabla Q_{m,3}(-1) &= \nabla([s_3]_{m,3}s_3 + [s_{21}]_{m,3}s_{21} + [s_{1^3}]_{m,3}s_{1^3}) \\ &= [s_3]_{m,3}\nabla s_3 + [s_{21}]_{m,3}\nabla s_{21} + [s_{1^3}]_{m,3}\nabla s_{1^3}. \end{aligned}$$

One can directly calculate that

$$\begin{aligned} &\nabla s_3 = (qt)^2 s_{21} + (qt)^2[2]_{q,t}s_{1^3}, \qquad \nabla s_{21} = -(qt)[2]_{q,t}s_{21} - (qt)[3]_{q,t}s_{1^3}, \\ &\nabla s_{1^3} = s_3 + ([2]_{q,t} + [3]_{q,t})s_{21} + (qt + [4]_{q,t})s_{1^3}, \end{aligned}$$

and the base cases of Theorem 4 are $Q_{1,3}(-1)$, $Q_{2,3}(-1)$ and $Q_{3,3}(-1)$, which can be verified by breaking the $Q_{m,3}$ operators into D_k operators and applying symmetric function manipulations. With the base cases verified, one prove Theorem 4 by inducting on m.

2.2 Combinatorial Side — $\mathrm{H}_{m,3}[X;q,t]$

Hikita [15] in 2012 proved that the Hikita polynomials $\mathrm{H}_{m,n}[X;q,t]$ are symmetric (in X) for any coprime m,n.

A weak composition of n is a sequence of non-negative integers summing up to n. Suppose that $\gamma = (\gamma_1,\ldots,\gamma_n)$ is a weak composition of n into n parts. We let $X = (x_1,\ldots,x_n)$, and $\Delta_\gamma(X) = \det ||x_i^{\gamma_j+n-j}|| = \sum_{\sigma\in S_n} sgn(\sigma)(x_{\sigma(1)}^{\gamma_1+n-1}\cdots x_{\sigma(n)}^{\gamma_n+n-n})$. Let $\Delta(X) = \det ||x_i^{n-j}||$ be the Vandermonde determinant, then the Schur function $s_\gamma(X)$ associated to γ is defined to be $s_\gamma(X) = \frac{\Delta_\gamma(X)}{\Delta(X)}$. It is well known that for any such weak composition γ, either $s_\gamma(X) = 0$ or there is a partition λ of n such that $s_\gamma(X) = \pm s_\lambda(X)$. In fact, there is a well-known straightening relation which allows on to prove that fact. Namely, if $\gamma_{i+1} > 0$, then $s_{(\gamma_1,\ldots,\gamma_i,\gamma_{i+1},\ldots,\gamma_n)}(X) = -s_{(\gamma_1,\ldots,\gamma_{i+1}-1,\gamma_i+1,\ldots,\gamma_n)}(X)$.

Suppose $\alpha = (\alpha_1,\ldots,\alpha_k)$ is a composition of n with k parts. We associate a subset $S(\alpha)$ of $\{1,\ldots,n-1\}$ with α by setting $S(\alpha) = \{\alpha_1,\alpha_1+\alpha_2,\ldots,\alpha_1+\cdots+\alpha_{k-1}\}$. We let $\tilde{\alpha}$ be the weak composition of n with n parts by adding a sequence of $n-k$ 0's at the end of α. For example, if $\alpha = (2,3,2,1)$, then $S(\alpha) = \{2,5,7\}$ and $\tilde{\alpha} = (2,3,2,1,0,0,0,0)$.

In a remarkable and important paper, Egge, Loehr and Warrington [5] gave a combinatorial description of how to start with the quasi-symmetric function expansion of a homogeneous symmetric function of degree n, $P(X) = \sum_{\alpha\Vdash n} a_\alpha F_\alpha(X)$ and transform it into an expansion in terms of Schur functions $P(X) = \sum_{\lambda\vdash n} b_\lambda s_\lambda(X)$. The following theorem due to Garsia and Remmel [6] is implicit in the work of [5], but is not explicitly stated and it allows one to find the Schur function expansion by using the straightening laws.

Theorem 7. *Suppose that $P(X) = \sum_{\alpha\Vdash n} a_\alpha F_\alpha(X)$ is a symmetric function which is homogeneous of degree n, then $P(X) = \sum_{\alpha\Vdash n} a_\alpha s_{\tilde{\alpha}}(X)$.*

Let $\mathrm{pides}(\sigma)$ be the composition set of $\mathrm{ides}(\sigma)$, then Theorem 5 and the straightening action allow us to transform $\mathrm{H}_{m,n}[X;q,t]$ into Schur function expansion that

$$\mathrm{H}_{m,n}[X;q,t] = \sum_{\mathrm{PF}\in\mathcal{PF}_{m,n}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}}[X] = \sum_{\mathrm{PF}\in\mathcal{PF}_{m,n}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} s_{\mathrm{pides(PF)}}.$$

Any parking function $\mathrm{PF} \in \mathcal{PF}_{m,3}$ has 3 rows, thus has only 3 cars: 1, 2, 3, so the word $\sigma(\mathrm{PF})$ can be any permutation $\sigma \in \mathcal{S}_3$. Table 1 shows the s_{pides} contribution of the 6 permutations in $\mathcal{S}_3$.

$\sigma \in \mathcal{S}_3$	123	132	213	231	312	321
s_{pides}	s_3	s_{21}	$s_{12} = 0$	s_{21}	$s_{12} = 0$	s_{1^3}

Table 1: s_{pides} contribution of the permutations in $\mathcal{S}_3$

By our notation, $\mathrm{H}_{m,3}[X;q,t] = [s_3]_{m,3}s_3 + [s_{21}]_{m,3}s_{21} + [s_{1^3}]_{m,3}s_{1^3}$. For an example of how one can work out the combinatorial side of rational shuffle conjecture in the case where $n = 3$, we shall briefly describe how one can use Theorem 7 to compute $[s_3]_{3k+1,3}$.

From Table 1, we see that only parking functions in $\mathcal{PF}_{3k+1,3}$ with a word 123 contribute to the coefficient of s_3. We also notice that the 3 cars should be in different columns, otherwise there are cars $i < j$ with $\mathrm{rank}(i) < \mathrm{rank}(j)$, contradicting that the word of the parking function is 123. Thus we have one PF $\in \mathcal{PF}_{3k+1,3}$ with word 123 on each $(3k+1,3)$ Dyck path which has no two consecutive north steps.

Let $\lambda(\mathrm{PF}) = \{\lambda_1, \lambda_2\}$ be the partition associated with the Dyck path $\Pi(\mathrm{PF})$, then we can count both area and dinv from $\lambda(\mathrm{PF})$. We have $\mathrm{area}(\mathrm{PF}) = 3k - \lambda_1 - \lambda_2$, and we can also write the formula for dinv:

$$\mathrm{dinv}(\mathrm{PF}) = \begin{cases} \lambda_1 - 2 & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \leq k, \\ 2\lambda_1 - k - 3 & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \geq k+1, \\ 2\lambda_2 + k - 2 & \text{if } \lambda_2 \geq 1 \text{ and } \lambda_1 - \lambda_2 \geq k+1. \end{cases}$$

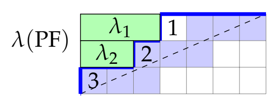


Figure 3: Example: a PF $\in \mathcal{PF}_{7,3}$ with word 123

For $[s_3]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t}$, we construct each term $(qt)^{k-1-i}[3i+1]_{q,t}$ as a sequence of parking functions. For each i, we have 3 branches of partitions (parking functions) to obtain $(qt)^{k-1-i}[3i+1]_{q,t}$:

$$\begin{aligned} \Lambda_1 &= \{(k+i+1,k), (k+i,k-1), \ldots, (k+1,k-i)\}, \\ \Lambda_2 &= \{(2k,i), (2k-1,i-1), \ldots, (2k+1-i,1)\}, \\ \Lambda_3 &= \{(k+1,i+1), (k,i+1), \ldots, (i+2,i+1)\}. \end{aligned}$$

The branch Λ_1 contains λ's such that $\lambda_1 - \lambda_2 = i+1 \leq k$ with $\lambda_2 \geq i+1$, the branch Λ_2 contains all λ's such that $\lambda_1 - \lambda_2 = 2k - i > k$, and the branch Λ_3 contains λ's such that $\lambda_2 = i+1$ and $\lambda_1 - \lambda_2 \leq k - i$. Notice that $|\Lambda_1| = |\Lambda_2| + 1$, and the last partition of Λ_1 is the same as the first partition in Λ_3. So as shown in Figure 4, the construction begins with *alternatively* taking partitions from Λ_1 and Λ_2, ending with the last partition of Λ_1. Then continue the chain by taking partitions in Λ_3 and end the chain with the last partition $(k-i+1, k-i)$ in Λ_3. Figure 5 shows the combinatorics of $[s_3]_{13,3}$.

We can combinatorially prove $[s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}([3i+2]_{q,t} + [3i+3]_{q,t})$ in a similar way. In this case, we have 2 possible diagonal words: 132 and 312, we can obtain 6 branches of parking functions for each i from 0 to $k-1$ based on the diagonal words and the shape of their partitions. 3 of the branches contribute $(qt)^{k-1-i}[3i+2]_{q,t}$ and the rest contribute $(qt)^{k-1-i}[3i+2]_{q,t}$ to the coefficient of s_{21}.

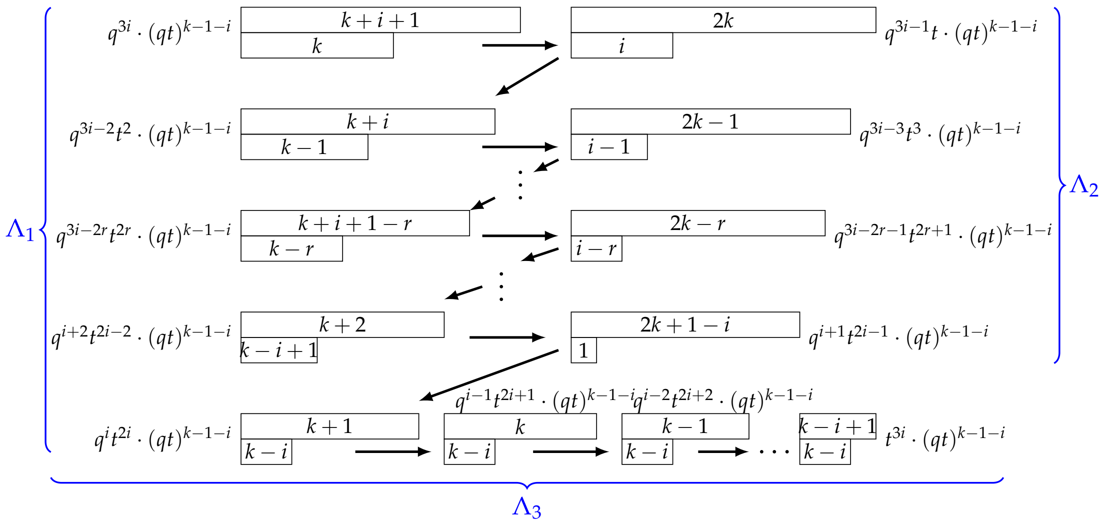


Figure 4: The construction of $(qt)^{k-1-i}[3i+1]_{q,t}$

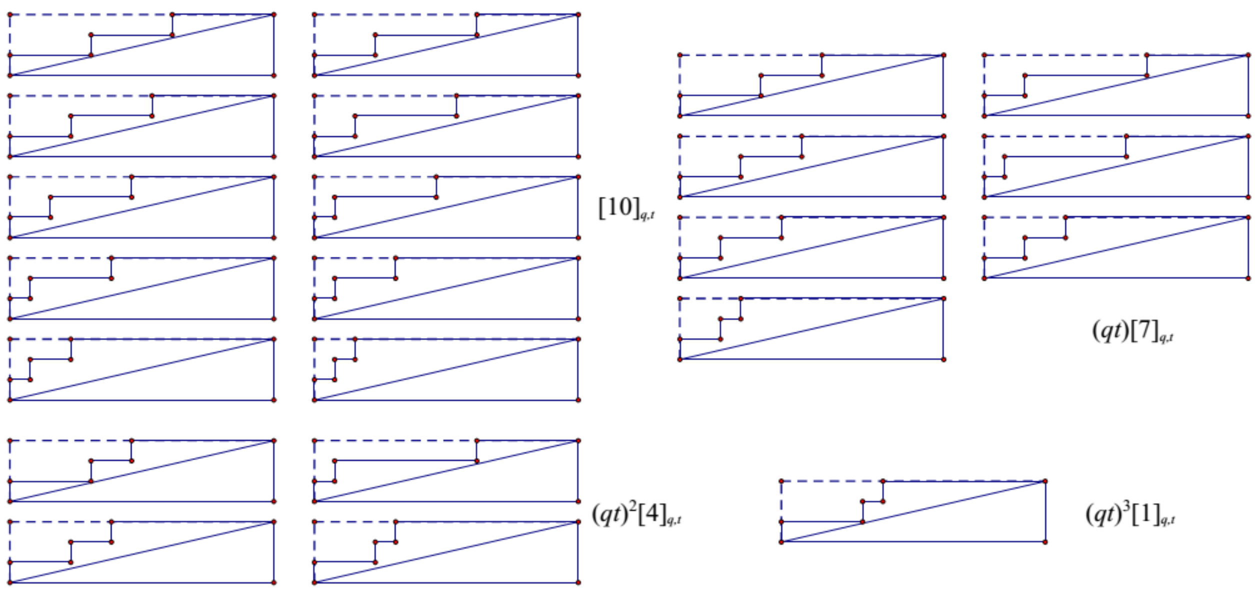


Figure 5: The combinatorics of $[s_3]_{13,3} = [10]_{q,t} + (qt)[7]_{q,t} + (qt)^2[4]_{q,t} + (qt)^3[1]_{q,t}$.

To prove that $[s_{1^3}]_{3k+1,3} = \sum_{i=0}^{k}(qt)^{k-i}[3i+1]_{q,t} = [s_3]_{3k+4,3}$, we can give a combinatorial proof of a more general result, namely, $[s_{1^3}]_{m,3} = [s_3]_{m+3,3}$.

Note that a parking function with pides can be straightened to 1^3 must have word 321. One PF $\in \mathcal{PF}_{m,3}$ with word 321 is correspond with an $(m,3)$ Dyck path. As shown is Figure 6, we can obtain a PF $\in \mathcal{PF}_{m+3,3}$ with word 123 by pushing a staircase into a PF $\in \mathcal{PF}_{m,3}$ with word 321. The fillings of the cars are fixed by the pides respectively.

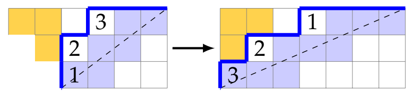


Figure 6: Bijection between $\mathcal{PF}_{m,3}$ with word 321 and $\mathcal{PF}_{m+3,3}$ with word 123

3 Combinatorial Results about $[s_\lambda]_{3,n}$

In $(3,n)$ case, we have n cars, i.e. the word of a $(3,n)$ parking function is a permutation of $[n]$. First, we can prove the following.

1. Let $i < j$ be two cars in the parking function. If i appears to the left of j in the diagonal word, then the cars i, j must be in different columns.
2. The parts in the composition set pides(PF) of a parking function $\mathrm{PF} \in \mathcal{PF}_{m,n}$ are less than or equal to m.

Thus, in the $(3,n)$ case, $[s_\lambda]_{3,n} \neq 0$ implies that $\lambda \vdash n$ must be of the form $3^a2^b1^c$; $[s_\lambda]_{m,n} \neq 0$ only if the partition λ only has parts of size less than or equal to m.

We can give bijective proofs of the following results. In each case, we shall briefly describe the idea how the bijection works.

Result. $[s_{3^{a+1}2^b1^c}]_{3,n} = [s_{3^a2^b1^c}]_{3,n-3}$.

$\mathrm{PF} \in \mathcal{PF}_{3,n}$ with pides $3^{a+1}2^b1^c$ must have the cars $1,2,3$ placed at the bottom of 3 columns in a rank decreasing way. The bijection is that we can remove these 3 cars and delete the corresponding north steps of the Dyck path to obtain a parking function of size $(3, n-3)$ with cars labeled from 4 to n; then we subtract 3 from the labels to obtain a $\mathrm{PF}' \in \mathcal{PF}_{3,n-3}$ with pides $3^a2^b1^c$, as Figure 7(a) shows. Clearly, the area is unchanged since the diagonal cells are still the previous diagonal cells, and the dinv statistic is also not changed since this manipulation keeps the rank of all the unremoved cars. This tells us that we only need to consider the coefficients $[s_{2^a1^b}]_{3,n}$.

Result. $[s_{1^3}]_{n,3} = [s_{1^n}]_{3,n}$.

The bijection for this identity is that we can transpose the path of $\mathrm{PF} \in \mathcal{PF}_{n,3}$ and fill the word $(n, n-1, \ldots, 1)$ to get PF'. It's easy to verify that we get $\mathrm{PF}' \in \mathcal{PF}_{3,n}$ with same area and dinv. Figure 7(b) shows an example of this bijection.

Result. $[s_{21}]_{n,3} = [s_{21^{n-2}}]_{3,n}$.

The bijective proof of this result is similar to the previous result. That is, one transposes the path and and labels the path to produce pides 21^{n-2}. If there are only 2 peaks in the Dyck path, then the filling of cars in both $(n,3)$ and $(3,n)$ cases are unique. Otherwise, in any rational $(n,3)$-Dyck path with 3 peaks, there are 2 kinds of words: 132 and 312 having pides 21 in $(n,3)$ case, which means that there are 2 choices to locate the car 1 in the $(3,n)$ case. The words that have pides 21^{n-2} must be of the form

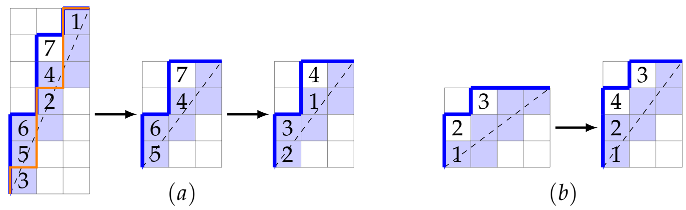


Figure 7: Bijection $[s_{3^{a+1}2^b1^c}]_{3,n} \longrightarrow [s_{3^a2^b1^c}]_{3,n-3}$ and bijection $[s_{1^3}]_{n,3} \longrightarrow [s_{1^n}]_{3,n}$

$(n, n-1, \ldots, i+1, 1, i, i-1, \ldots, 2)$, so the car 1 can be placed at the bottom of either the second column or the third column. We are able to match the 2 possible positions of car 1 in both $(n, 3)$ and $(3, n)$ cases by the dinv statistic, thus prove the result.

Result. $[s_{2^a1^b}]_{3,n} = [s_{2^b1^a}]_{3,3(a+b)-n}$.

We have found the straightening action in parking functions combinatorially from pides$\{\cdots 1, 3 \cdots\}$ to pides$\{\cdots 2, 2 \cdots\}$, which is an *involution* whose fixed points are the coefficients of $[s_{2^a1^b}]_{3,n}$. Further we have found a bijection between the fixed parking functions with pides 2^a1^b and the fixed parking functions with pides 2^b1^a, mapping the 2 cars (or 1 car) causing part 2 (or 1) in pides 2^a1^b to 1 car (or 2 cars) causing part 1 (or 2) in pides 2^b1^a.

The four results above prove Theorem 5. Finally, we conjecture a recursive formula for $[s_{2^a1^b}]_{3,n}$.

Conjecture 1. *Let $a < b$, then $[s_{2^a1^b}]_{3,n} = \sum_{i=0}^{a}[b+i]_{q,t} + (qt)[s_{2^a1^{b-3}}]_{3,n-3}$.*

We verified this formula by Maple for $n < 27$. If this conjecture is true, then we have solved the Schur function expansion in the $(3, n)$ case.

4 Combinatorial Results about $[s_\lambda]_{3,n}$

Using the ideas of the previous section, we can give bijective proofs of the following general results.

Theorem 6. *For all $m, n > 0$ and $\lambda' \vdash (n - am)$,* **(a)** $[s_{1^n}]_{m-n,n} = [s_n]_{m,n}$, **(b)** $[s_{m^a\lambda'}]_{m,n} = [s_{\lambda'}]_{m,n-am}$, **(c)** $[s_{1^n}]_{m,n} = [s_{1^m}]_{n,m}$, *and* **(d)** $[s_{k1^{n-k}}]_{m,n} = [s_{k1^{m-k}}]_{n,m}$.

We haven't completely understood how to use straightening to compute the coefficients of s_λ for general (m, n) case, but computations in Maple have led us to conjecture the following.

Conjecture 2. $[s_{(m-1)^{\alpha_{m-1}}(m-2)^{\alpha_{m-2}}\ldots 1^{\alpha_1}}]_{m,n} = [s_{(m-1)^{\alpha_1}(m-2)^{\alpha_2}\ldots 1^{\alpha_{m-1}}}]_{m,(m\sum_{i=1}^{m-1}\alpha_i - n)}$.

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